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Approximation Numbers of Sobolev Imbeddings over Unbounded Domains

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Let $\Omega \subset \mathbb{R}^N$ be an open set with $\operatorname{dist}(x, \partial\Omega) = O(|x|^{-l})$ for $x \in \Omega$ and some $l > 0$ satisfying an additional regularity condition. We give asymptotic estimates for the approximation numbers α_n of Sobolev imbeddings $I^m_{p,q} : \mathring{W}_p{}^m(\Omega) \to L_q(\Omega)$ over these quasibounded domains Ω. Here $\mathring{W}_p{}^m(\Omega)$ denotes the Sobolev space obtained by completing $C_0{}^\infty(\Omega)$ under the usual Sobolev norm. We prove

$$\alpha_n(I^m_{p,q}) \lesssim n^{-\gamma},$$

where

$$\begin{aligned} \gamma &= \frac{1}{l+1}\left(\frac{m}{N} + \frac{1}{l}\left(\frac{1}{p} - \frac{1}{q}\right)\right) && \text{for} \quad p \geqslant q, \quad ml > N\left(\frac{1}{q} - \frac{1}{p}\right) \\ &= \frac{1}{l+1}\left(\frac{m}{N} + \left(\frac{1}{q} - \frac{1}{p}\right)\right) && \text{for} \quad p \leqslant q, \quad m > N\left(\frac{1}{p} - \frac{1}{q}\right). \end{aligned}$$

There are quasibounded domains of this type where γ is the exact order of decay, in the case $p \leqslant q$ under the additional assumption that either $1 \leqslant p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q \leqslant \infty$. This generalizes the known results for bounded domains which correspond to $l = \infty$. Similar results are indicated for the Kolmogorov and Gelfand numbers of $I^m_{p,q}$. As an application we give the rate of growth of the eigenvalues of certain elliptic differential operators with Dirichlet boundary conditions in $L_2(\Omega)$, where Ω is a quasibounded domain of the above type.

1. INTRODUCTION

Let Ω be an open set in $\mathbb{R}^N$ and $C^\infty(\Omega)[C_0{}^\infty(\Omega)]$ be the space of infinitely differentiable functions on Ω [with compact support in Ω]. Let $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. By $W_p{}^m(\Omega)$ $[\mathring{W}_p{}^m(\Omega)]$ we denote the Sobolev space obtained by completing $C^\infty(\Omega)$ $[C_0{}^\infty(\Omega)]$ under the norm

$$\|f\|_{W_p{}^m} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha f(x)|^p \, dx\right)^{1/p},$$

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with the usual interpretation for $p = \infty$. We also introduce a seminorm on $W_p^m(\Omega)$,

$$|f|_{W_p^m} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha f(x)|^p \, dx \right)^{1/p}.$$

An open set $\Omega \subset \mathbb{R}^N$ is called quasibounded if and only if $\lim_{n\to\infty} \sup_{x\in\Omega_n} \tau(x) = 0$, where $\tau(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Omega_n := \Omega \cap \{x \in \mathbb{R}^N : \|x\|_\infty := \sup_{1\leqslant i\leqslant N} |x_i| > n\}$ cf. [5].

We say that Ω satisfies condition C_k^l, with $l \in \mathbb{R}^+$ and $k \in \mathbb{N}$, $1 \leqslant k \leqslant N$, if there is a monotone non-increasing function $d: \mathbb{R}^+ \to \mathbb{R}^+$ fulfilling $d(n) = O(n^{-l})$ for $n \to \infty$ and if there are constants $c, n_0 \in \mathbb{R}^+$ such that for any $n \geqslant n_0$ and any cube H of side length $d(n)$ intersecting Ω_n one has $\mu_{N-k}(H, \Omega) \geqslant cd(n)^{N-k}$, where $\mu_{N-k}(H, \Omega)$ denotes the maximum of the $(N-k)$-measure of $P(H \sim \Omega)$, taken over all projections P onto $(N-k)$-dimensional faces of H.

Condition C_k^l implies quasiboundedness. Let $\Omega \subset \mathbb{R}^N$ be an open set with

(1) $\sup_{x\in\Omega_n} \tau(x) = O(n^{-l})$,

(2) there is $m \in \mathbb{N}$, $1 \leqslant m \leqslant \mathbb{N}$, and an m-dimensional cone C with vertex in 0, such that $\mathbb{R}^N \sim \Omega$ contains with every x an m-dimensional cone C_x with vertex x congruent to C.

Then Ω fulfills condition C_{N-m}^l for $m < N$ and C_1^l for $m = N$; cf. [10].

Let X and Y be normed linear spaces and $\mathscr{L}(X, Y)$ denote the continuous linear operators from X into Y and let $T \in \mathscr{L}(X, Y)$. The approximation numbers of T are defined for $n \in \mathbb{N}$ by

$$\alpha_n(T) := \inf\{\|T - T_n\| : \operatorname{rank} T_n \leqslant n\}, \qquad \alpha_0(T) := \|T\|.$$

The Kolmogorov numbers of T are given by

$$\beta_n(T) := \inf\{\sup\{\operatorname{dist}(Tx, L_n) : \|x\| \leqslant 1\}\},$$

where the infimum is taken over all subspaces L_n of Y with $\dim L_n \leqslant n$. The Gelfand numbers are given by

$$\gamma_n(T) := \inf\{\|T|_M\| : \operatorname{codim} M \geqslant n\}.$$

The α_n, β_n, γ_n are examples of s-number sequences. For properties of these s-numbers we refer to Pietsch [13]. The approximation numbers, Kolmogorov numbers and Gelfand numbers of Sobolev imbeddings of type $W_p^m(\Omega) \to L_q(\Omega)$ or $\mathring{W}_p^m(\Omega) \to L_q(\Omega)$ have been estimated asymptotically for $n \to \infty$ in the case of a bounded domain Ω. See Ismagilov [8] for the known results and a bibliography. We refer further to [4, 14]. For the approximation numbers, only in the case $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ the exact order of decay seems to be unknown.

We will generalize these results for imbeddings of type $\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)$ to quasibounded domains $\Omega \subset \mathbb{R}^N$. In general very little can be said about imbeddings $W_p{}^m(\Omega) \to L_q(\Omega)$, since these imbeddings are mostly not even compact; cf. Adams and Fournier [2].

If δ_n and ϵ_n are sequences of positive numbers, we write $\delta_n \lesssim \epsilon_n$, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$: $\delta_n \leqslant c\epsilon_n$. We use the notation $\delta_n \sim \epsilon_n$ for $\epsilon_n \lesssim \delta_n$ and $\delta_n \lesssim \epsilon_n$.

Section 2 contains the results on the s-numbers of imbeddings of the above type, and Section 3 contains an application to elliptic differential operators.

2. s-Numbers of Sobolev Imbeddings

THEOREM 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set fulfilling condition $C_k{}^l$ with $l \in \mathbb{R}^+$, $k \in \mathbb{N}$ and $1 \leqslant k \leqslant N$. Let $1 \leqslant p, q \leqslant \infty$ with $p > k$. Then*

(a) *If $p \geqslant q$ and $ml > N(1/q - 1/p)$, the imbedding $\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)$ exists and*

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-\delta}, \qquad \delta = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{l}\left(\frac{1}{p} - \frac{1}{q}\right)\right).$$

(b) *If $p \leqslant q$ and $m > N(1/p - 1/q)$, the imbedding $\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)$ exists and*

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-\delta}, \qquad \delta = \frac{l}{l+1}\left(\frac{m}{N} + \left(\frac{1}{q} - \frac{1}{p}\right)\right),$$

(c) *There are quasibounded domains $\Omega \subset \mathbb{R}^N$ satisfying condition $C_k{}^l$ such that*

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \sim n^{-\delta}, \qquad \delta = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{l}\left(\frac{1}{p} - \frac{1}{q}\right)\right)$$

in case (a) *and*

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \sim n^{-\delta}, \qquad \delta = \frac{l}{l+1}\left(\frac{m}{N} + \left(\frac{1}{q} - \frac{1}{p}\right)\right)$$

in case (b), *if additionally either $1 \leqslant p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q \leqslant \infty$.*

Remark. The case of a bounded domain is formally included for $l = \infty$, $l/(l+1)$ being replaced by 1. In case (a), the expression $(l+1)^{-1}(1/q - 1/p)$ in the exponent is a compensation factor, since for unbounded domains and $p \geqslant q$ no longer $L_p(\Omega) \subset L_q(\Omega)$. In general, the approximation numbers tend slower to zero, if the "degree of unboundedness" $1/l$ of Ω increases, i.e. if l gets smaller. The most interesting case for k is $k = 1$, if Ω satisfies condition $C_1{}^l$, there is no restriction on p except $p > 1$.

LEMMA 1. *Let $\nu \in \mathbb{N}$ and $p > N$. There is a constant $c > 0$ such that for any $h > 0$, any cube H of side h in $\mathbb{R}^N$ and all $f \in C^\infty(H)$ which vanish in a neighborhood of a point $y \in H$,*

$$\|f\|_{L_p(H)} \leqslant ch^\nu \,|f|_{W_p^\nu(H)} .$$

This is Proposition 1 of [10], the norm $\|f\|_{W_p^\nu(H)}$ there can be replaced by $|f|_{W_p^\nu(H)}$, as the proof shows. We need, further, Lemma 1 of Adams [1]:

LEMMA 2. *There is a constant c depending on p and N such that for any $h > 0$, any cube H of side h in $\mathbb{R}^N$, any set $A \subset H$ with positive measure $\lambda(A) > 0$ and any $f \in C^\infty(H)$*

$$\|f\|_{L_p(H)} \leqslant c\left(\frac{h^N}{\lambda(A)}\right)^{1/p}\left(\|f\|_{L_p(A)} + h\sum_{|\alpha|=1}\|D^\alpha f\|_{L_p(H)}\right).$$

We need a Poincare-type inequality.

PROPOSITION 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set fulfilling condition C_k^l with function d, $l \in \mathbb{R}^+$, $1 \leqslant k \leqslant N$. Let $1 \leqslant p, q \leqslant \infty$ and $m \in \mathbb{N}$ with $p > k$ and $m > \max(0, N(1/p - 1/q))$. Let H_n denote any cube of side $d(n)$ in $\mathbb{R}^N$ with faces parallel to the coordinate axes intersecting Ω_n. Then there is $n_0 \in \mathbb{N}$ and a constant $c > 0$ depending on Ω, m, p and q such that for any $n \geqslant n_0$, any H_n and any $g \in C_0^\infty(\Omega)$,*

$$\|g\|_{L_q(H_n)} \leqslant cd(n)^{m+N(1/q-1/p)}\,|g|_{W_p^m(H_n)} \tag{2.1}$$

where g is assumed to be extended by zero outside of Ω.

Proof of Proposition 1. (a) We consider first the case $p = q$. For the sake of completeness we give the full argument, although similar proofs have been shown in [1, 10]. So let $p = q$, $n \geqslant n_0$ and H_n be a cube of side $d(n)$ with faces parallel to the axes such that $H_n \cap \Omega_n \neq \varnothing$. In the following, $c_1, c_2, \ldots$ are (positive) constants not depending on n, H_n, $d(n)$ and $f \in C_0^\infty(\Omega)$. By assumption on Ω, $\mu_{N-k}(H_n, \Omega) \geqslant c_1 d(n)^{N-k}$. Let P be the maximal projection referred to in the μ_{N-k}-definition and $E = P(H_n \sim \Omega)$. We may assume that the $(N-k)$-face F of H_n containing E is parallel to the $x_{k+1}, \ldots, x_N$-axes. For $x = (x', x'')$ in E with $x' = (x_1, \ldots, x_k)$, $x'' = (x_{k+1}, \ldots, x_N)$ let $H_{x''}$ be the k-cube in which H_n intersects the k-plane through x normal to F. Then there is $y \in H_{x''} \sim \Omega$. For $f \in C_0^\infty(\Omega)$, $f(\cdot, x'') \in C^\infty(H_{x''})$ vanishes in a (k-dimensional) neighborhood of $y \in H_{x''}$. By Lemma 1 there is $c_2 > 0$ independent of x'' and $\nu \in \mathbb{N}$, $1 \leqslant \nu \leqslant m$ such that

$$\left(\int_{H_{x''}} |f(x', x'')|^p\,dx'\right) \leqslant c_2 d(n)^{\nu p}\left(\sum_{|\alpha|=\nu}\int_{H_{x''}} |D_{x'}^\alpha f(x', x'')|^p\,dx'\right).$$

Let $H' = \{x' : x = (x', x'') \in H_n \text{ for some } x''\}$. Integration over E yields

$$\begin{aligned} \|f\|_{L_p(H'\times E)} &\leqslant c_3 d(n)^\nu \,|f|_{W_p^\nu(H'\times E)} \\ &\leqslant c_3 d(n)^\nu \,|f|_{W_p^\nu(H_n)} \end{aligned} \tag{2.2}$$

Let $A = H' \times E$. Then $\lambda(A) = d(n)^k \cdot \mu_{N-k}(H_n, \Omega)$ and an m-fold application of Lemma 2 yields for every $g \in C_0^\infty(\Omega)$,

$$\|g\|_{L_p(H_n)} \leqslant c_4 \left(\sum_{j=0}^{m-1} d(n)^j \left[\sum_{|\alpha|=j} \|D^\alpha g\|_{L_p(A)} \right] + d(n)^m \,|g|_{W_p^m(H_n)} \right) \tag{2.3}$$

We used here $(d(n)^N/\lambda(A))^{1/p} \leqslant c_1^{-1/p} < \infty$ m times. We apply (2.2) to each of the $D^\alpha g$'s in (2.3) with $|\nu| = m - |\alpha|$, to find $c_5 > 0$ such that

$$\|g\|_{L_p(H_n)} \leqslant c_5 d(n)^m \,|g|_{W_p^m(H_n)}\,.$$

This is (2.1) for $p = q$.

(b) Assume now $p \neq q$. Let $Q = \{x \in \mathbb{R}^N : \|x\|_\infty \leqslant 1\}$. Since $m > \max(0, N(1/p - 1/q))$, the Sobolev imbedding $W_p^m(Q) \to L_q(Q)$ exists and there is $c_6 > 0$ such that for all $f \in C^\infty(Q)$,

$$\|f\|_{L_q(Q)} \leqslant c_6 \|f\|_{W_p^m(Q)}\,. \tag{2.4}$$

Let $n \geqslant n_0$ and H_n be as above, $H_n \cap \Omega_n \neq \varnothing$. If y_0 is the midpoint of H_n, $Fg(x) = g(y_0 + d(n)x)$ defines a bijection $F: C^\infty(H_n) \to C^\infty(Q)$. We get by substitution and (2.4)

$$\begin{aligned} \|g\|_{L_q(H_n)} &= d(n)^{N/q} \|Fg\|_{L_q(Q)} \\ &\leqslant c_6 d(n)^{N/q} \|Fg\|_{W_p^m(Q)} \end{aligned}$$

for any $g \in C_0^\infty(\Omega)$ which is extended by zero outside of Ω. Again by substitution,

$$\|g\|_{L_q(H_n)} \leqslant c_6 d(n)^{N(1/q-1/p)} \left(\sum_{|\alpha| \leqslant m} d(n)^{|\alpha| p} \int_{H_n} |D_y^\alpha g(y)|^p \, dy \right)^{1/p}. \tag{2.5}$$

Since also $D^\alpha g \in C_0^\infty(\Omega)$, we may apply the for $p = q$ already known formula (2.1) to $D^\alpha g$, for all α with $|\alpha| \leqslant m$ and for $m - |\alpha|$ instead of m to get from (2.5)

$$\begin{aligned} \|g\|_{L_q(H_n)} &\leqslant c_7 d(n)^{N(1/q-1/p)} \left(\sum_{|\beta|+|\alpha|=m} d(n)^{(|\beta|+|\alpha|)p} \int_{H_n} |D^\beta D^\alpha g(y)|^p \, dy \right)^{1/p} \\ &= c_7 d(n)^{m+N(1/q-1/p)} \,|g|_{W_p^m(H_n)} \end{aligned}$$

for any $g \in C_0^\infty(\Omega)$, any $n \geqslant n_0$ and any H_n, $H_n \cap \Omega_n \neq \varnothing$. ▮

LEMMA 3. *Let* $Q = \{x \in \mathbb{R}^N : \| x \|_\infty \leqslant 1\}$. *Let* P_n *denote the* nth *trigonometric projection,*

$$P_n f = \sum_{k_1, \ldots, k_N = -n}^{n} (f, g_k) g_k, \qquad f \in L_1(Q), \quad g_k(t) = e^{2\pi i k \cdot t}.$$

Assume $1 < p < q < \infty$ *and* $m > N(1/p - 1/q)$. *Then there is* $c > 0$ *such that for any* $f \in W_p{}^m(Q)$,

$$\| f - P_n f \|_{L_q(Q)} \leqslant c n^{-(m+N(1/q-1/p))} \| f \|_{W_p{}^m(Q)} .$$

Proof. Since $m > N(1/p - 1/q)$, any $f \in W_p{}^m(Q)$ is in $L_q(Q)$. Consider the operator $F_n : L_p(Q) \to L_q(Q)$ of convolution with $h_n = \sum_{k_1, \ldots, k_N = -n}^{n} g_k$. By Young's inequality,

$$\| F_n \| \leqslant \| h_n \|_{L_s(Q)}, \qquad 1/s = 1 - 1/p + 1/q.$$

But

$$\| h_n \|_{L_s(Q)} = \prod_{i=1}^{n} \Big\| \sum_{k_i = -n}^{n} g_{k_i} \Big\|_{L_s(0,1)} \sim n^{N(1/p-1/q)},$$

for the last estimate cf. Ismagilov [8]. Hence for any trigonometric polynomial f_n of order $\leqslant n$ in any variable,

$$\| f_n \|_{L_q(Q)} = \| h_n * f_n \|_{L_q(Q)} \leqslant c_1 n^{N(1/p-1/q)} \| f_n \|_{L_p(Q)} \tag{2.6}$$

By Johnen [9] there is $c_2 > 0$ such that for any $n \in \mathbb{N}$ and any $f \in W_p{}^m(Q)$,

$$\| P_{2^{n+1}} f - P_{2^n} f \|_{L_p(Q)} \leqslant c_2 (2^n)^{-m} \| f \|_{W_p{}^m(Q)},$$

a Jackson-type inequality. This yields together with (2.6)

$$\| P_{2^{n+1}} f - P_{2^n} f \|_{L_q(Q)} \leqslant c_3 (2^n)^{-(m+N(1/q-1/p))} \| f \|_{W_p{}^m(Q)}$$

Since $1 < q < \infty$, the partial sums $P_{2^n} f$ converge to f in the $L_q(Q)$-norm for any $f \in L_q(Q)$; cf. Dunford-Schwartz [6, p. 368]. Hence

$$f - P_{2^n} f = \sum_{j=n}^{\infty} (P_{2^{n+1}} f - P_{2^\nu} f)$$

and

$$\begin{aligned} \| f - P_{2^n} f \|_{L_q(Q)} &\leqslant c_3 \sum_{j=n}^{\infty} (2^j)^{-\alpha} \| f \|_{W_p{}^m(Q)} \\ &= c_4 (2^n)^{-\alpha} \| f \|_{W_p{}^m(Q)} \end{aligned} \tag{2.7}$$

with $\alpha = m + N(1/q - 1/p) > 0$, using the geometric series. Lemma 3 follows immediately from (2.7). ∎

We are now ready to give the

Proof of Theorem 1. If Ω fulfills condition $C_k{}^l$, the Sobolev imbeddings I: $\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)$ exist under the conditions stated in (a) and (b). This follows from Theorem 1 of [10]. In the following, constants may depend on p, q, m and Ω, but not on n.

(1) Let $\phi(n) := (2^n)^{(N(l+1))^{-1}}$ and $Q_n := \{x \in \mathbb{R}^N : \| x \|_\infty \leqslant \phi(n)\}$. We define

$$E_n : \mathring{W}_p{}^m(\Omega) \to W_p{}^m(Q_n), \qquad E_n f(x) = \begin{cases} f(x), & x \in \Omega \cap Q_n, \\ 0, & \text{otherwise;} \end{cases}$$

$$R_n : L_q(Q_n) \to L_q(\Omega), \qquad R_n f(x) = \begin{cases} f(x), & x \in Q_n \cap \Omega, \\ 0, & \text{otherwise;} \end{cases}$$

$$J_n : \mathring{W}_p{}^m(\Omega) \to L_q(\Omega), \qquad J_n f(x) = \begin{cases} f(x), & x \in Q_n \cap \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Let I_n denote the Sobolev imbedding $W_p{}^m(Q_n) \to L_q(Q_n)$. Then $R_n \circ I_n \circ E_n = J_n$ and $\| E_n \| = \| R_n \| = 1$. By the properties of the approximation numbers (cf. [12, 13]), we have

$$\begin{aligned} \alpha_{2^n}(I) &\leqslant \alpha_{2^n}(J_n) + \| I - J_n \| \\ &\leqslant \alpha_{2^n}(I_n) + \| I - J_n \|. \end{aligned} \tag{2.8}$$

We estimate both terms differently. Let $Q = \{x \in \mathbb{R}^N : \| x \|_\infty \leqslant 1\}$ and define a bijection F: $C^\infty(Q_n) \to C^\infty(Q)$ by $Ff(x) = f(\phi(n)x)$, $f \in C^\infty(Q_n)$. F can be extended continuously to $W_p{}^m(Q_n)$ and $L_q(Q_n)$ with

$$\| Ff \|_{W_p{}^m(Q)} \leqslant \phi(n)^{m - N/p} \| f \|_{W_p{}^m(Q_n)}$$

and

$$\| f \|_{L_q(Q_n)} \leqslant \phi(n)^{N/q} \| Ff \|_{L_q(Q)},$$

as seen by substitution. If I_0: $W_p{}^m(Q) \to L_q(Q)$ is the Sobolev imbedding, we have

$$\begin{aligned} \alpha_{2^n}(I_n) &\leqslant \| F : W_p{}^m(Q_n) \to W_p{}^m(Q) \| \times \alpha_{2^n}(I) \times \| F^{-1} : L_q(Q) \to L_q(Q_n) \| \\ &\leqslant \phi(n)^{m + N(1/q - 1/p)} \alpha_{2^n}(I_0). \end{aligned}$$

Estimates for the approximation numbers of I_0 are known (cf. [4, 8]): If $p \geqslant q$,

$$\alpha_{2^n}(I_0) \leqslant c_1 (2^n)^{-m/N};$$

hence

$$\alpha_{2^n}(I_n) \leqslant c_2 (2^n)^{-\beta}, \qquad \beta = \frac{l}{l+1} \left(\frac{m}{N} + \frac{1}{l} \left(\frac{1}{p} - \frac{1}{q} \right) \right). \tag{2.9}$$

For $p \leqslant q$, $q \neq \infty$ it is an easy consequence of Lemma 3 that

$$\alpha_{2^n}(I_0) \leqslant c_2(2^n)^{-(m/N+1/q-1/p)},$$

and for $q = \infty$ this is also valid; cf. [14]. Hence

$$\alpha_{2^n}(I_n) \leqslant c_2(2^n)^{-\beta}, \qquad \beta = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{q} - \frac{1}{p}\right). \tag{2.10}$$

Therefore the first term on the right side of (2.8) has the order of decay stated in (a) and (b) of Theorem 1. We show the same for the second term $\| I - J_n \|$.

Let A_n be the domain $Q_{n+1} \sim Q_n$. Disregarding the boundaries of the cubes, $A_n \cap \Omega \subset \Omega_{\phi(n)}$ can be tesselated by $\psi(n)$ disjoint cubes of side $d(\phi(n))$, where

$$\psi(n) \leqslant (\phi(n+1)/d(\phi(n)))^N.$$

Let H_i, $i = 1,\ldots, \psi(n)$, denote these cubes. Then for any $g \in C_0^\infty(\Omega)$, extended by zero outside of Ω,

$$\| g \|_{L_q(A_n)} = \left(\sum_{i=1}^{\psi(n)} \| g \|_{L_q(H_i)}^q\right)^{1/q}.$$

We may assume $\phi(n) \geqslant n_0$, where n_0 is the value in Proposition 1. By Proposition 1, there is c_3 independent of n and g such that

$$\| g \|_{L_q(A_n)} \leqslant c_3 d(\phi(n))^{m+N(1/q-1/p)} \left(\sum_{i=1}^{\psi(n)} | g |_{W_p^m(H_i)}^q\right)^{1/q}. \tag{2.11}$$

For $p \geqslant q$ this is smaller than

$$\begin{aligned}
&\leqslant c_3 d(\phi(n))^{m+N(1/q-1/p)} \left(\sum_{i=1}^{\psi(n)} | g |_{W_p^m(H_i)}^p\right)^{1/p} \\
&\leqslant c_3 d(\phi(n))^{m+N(1/q-1/p)} \| g \|_{W_p^m(\Omega)} \\
&\leqslant c_4(2^n)^{-(l/(l+1))(m/N+1/q-1/p)} \| g \|_{W_p^m(\Omega)}.
\end{aligned} \tag{2.12}$$

Since by Hölder's inequality for $\xi_i \in \mathbb{C}$

$$\left(\sum_{i=1}^{\psi(n)} | \xi_i |^q\right)^{1/q} \leqslant \psi(n)^{1/q-1/p} \left(\sum_{i=1}^{\psi(n)} | \xi_i |^p\right)^{1/p},$$

we get for $p > q$ that the right side of (2.11) is smaller than

$$\begin{aligned}
&\leqslant c_3 d(\phi(n))^{m+N(1/q-1/p)} \psi(n)^{1/q-1/p} \| g \|_{W_p^m(\Omega)} \\
&= c_3 d(\phi(n))^m \phi(n+1)^{N(1/q-1/p)} \| g \|_{W_p^m(\Omega)} \\
&\leqslant c_5(2^n)^{-\gamma} \| g \|_{W_p^m(\Omega)}, \qquad \gamma = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{l}\left(\frac{1}{p} - \frac{1}{q}\right)\right).
\end{aligned} \tag{2.13}$$

Let $c_6 = \max(c_4, c_5)$. Equations (2.12) and (2.13) imply for any $g \in C_0^\infty(\Omega)$,

$$\|(I - J_n)g\|_{L_q(\Omega)} = \|g\|_{L_q(\Omega_{\phi(n)})} \leqslant \sum_{j=n}^{\infty} \|g\|_{L_q(A_j)}$$

$$\leqslant c_6 \sum_{j=n}^{\infty} (2^j)^{-\beta} \|g\|_{W_p^m(\Omega)}$$

$$\leqslant c_7 (2^n)^{-\beta} \|g\|_{W_p^m(\Omega)},$$

with

$$\beta = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{l}\left(\frac{1}{p} - \frac{1}{q}\right)\right) \quad \text{for} \quad p \geqslant q$$
$$= \frac{l}{l+1}\left(\frac{m}{N} + \left(\frac{1}{q} - \frac{1}{p}\right)\right) \quad \text{for} \quad p \leqslant q. \tag{2.14}$$

Hence $\|I - J_n\| \leqslant c_7(2^n)^{-\beta}$ and by (2.6), (2.7) and (2.8)

$$\alpha_{2^n}(I) \leqslant c(2^n)^{-\beta}.$$

This shows (a) and (b) of Theorem 1.

(2) We proceed to the proof of (c) of Theorem 1. For $j \in \mathbb{N}$ let $Q(j) = \{x \in \mathbb{R}^N : \|x\|_\infty < 2^j\}$ and $A(j) = Q(j+1) \sim Q(j)$. Let $l \in \mathbb{R}^+$. Since volume $A(j) \sim 2^{jN}$, $A(j)$ contains $\psi(j)$ disjoint open cubes H_{ij}, $i = 1,..., \psi(j)$ of side $(2^j)^{-l}$, with $\psi(j) \geqslant c_1(2^j)^N/(2^j)^{-lN} = c_1(2^j)^{N(l+1)}$ for some constant c_1 independent of j. Let $\Omega(j) = \bigcup_{i=1}^{\psi(j)} H_{ij}$ be the union of these $\psi(j)$ disjoint open cubes. Define $\Omega = \bigcup_{j=1}^{\infty} \Omega(j)$. It is easy to see that Ω fulfills condition C_k^l for any $1 \leqslant k \leqslant N$. We will show that

$$\alpha_n(I: \mathring{W}_p^m(\Omega) \to L_q(\Omega)) \gtrsim n^{-\beta},$$

β as in (2.14), for those values of p and q stated in (c). Let $Q = Q(0)$ and choose $\bar{f} \in C_0^\infty(Q)$, $\bar{f} \neq 0$ with $\|\bar{f}\|_{W_p^m(Q)} = 1$, $\|\bar{f}\|_{L_q(Q)} = M_1$, $\|\bar{f}\|_{L_\infty(Q)} = M_2$ and $\|\bar{f}\|_{L_2(Q)} = M$. If y_{ij} is the midpoint of H_{ij},

$$f_{ij}(x) = \bar{f}(2^{jl}(x - y_{ij}))$$

defines a function $f_{ij} \in C_0^\infty(H_{ij})$ with

$$\|f_{ij}\|_{W_p^m(H_{ij})} \leqslant (2^j)^{ml - Nl/p}, \qquad \|f_{ij}\|_{L_q(H_{ij})} = (2^j)^{-Nl/q} M_1 \tag{2.15}$$

Extend each f_{ij} to Ω by setting $f_{ij}(x) = 0$ for $x \in \Omega \sim H_{ij}$. We denote the $\psi(j)$-dimensional of $W_p^m(\Omega)$ $[L_q(\Omega)]$ spanned by the functions (f_{ij}), $i = 1,..., \psi(j)$, by $P_j W_p^m [P_j L_q]$ respectively. The spaces are isometric to $l_p^{\psi(j)}$ $[l_q^{\psi(j)}]$, since the supports of different f_{ij}'s are disjoint. We factor the identity map from $l_p^{\psi(j)}$ to $l_q^{\psi(j)}$ as

$$l_p^{\psi(j)} \xrightarrow{I_p} P_j W_p{}^m \xrightarrow{J_1} \mathring{W}_p{}^m(\Omega) \xrightarrow{J_2} L_q(\Omega) \xrightarrow{E} P_j L_q \xrightarrow{I_q^{-1}} l_q^{\psi(j)},$$

where J_1 and J_2 are the obvious inclusions, I_p is the map sending $(\lambda_i) \in l_p^{\psi(j)}$ to $\sum_{i=1}^{\psi(j)} \lambda_i f_{ij}$ and where E is the projection given by

$$Eg = (2^j)^{Nl} M^{-2} \sum_{i=1}^{\psi(j)} (g, f_{ij}) f_{ij}, \qquad g \in L_q(\Omega).$$

E is a projection because $(f_{i_1 j}, f_{i_2 j}) = 0$ for $i_1 \neq i_2$ and $(2^j)^{Nl} M^{-2} = (f_{ij}, f_{ij})^{-1}$. Easy estimates using (2.15) show

$$\| I_p \| \leqslant (2^j)^{ml - Nl/p}, \qquad \| I_q^{-1} \| = M_1^{-1} (2^j)^{Nl/q}.$$

Further, $\| E \| \leqslant M_1 M_1 / M^2$, since by (2.15) and Hölder's inequality for any $g \in L_q(\Omega)$,

$$\begin{aligned} \| Eg \|_{L_q(\Omega)} &\leqslant (2^j)^{Nl} M^{-2} \sum_{i=1}^{\psi(j)} |(g, f_{ij})| \, \| f_{ij} \|_{L_q(\Omega)} \\ &\leqslant (2^j)^{Nl/q'} M_1 M_2 M^{-2} \cdot \sum_{i=1}^{\psi(j)} \int_{H_{ij}} | g(x) | \, dx \\ &\leqslant M_1 M_2 M^{-2} \| g \|_{L_q(\Omega)} . \end{aligned}$$

Using the properties of the approximation numbers and the previous norm estimates, we get for any $n \in \mathbb{N}$

$$\alpha_n(Id\colon l_p^{\psi(j)} \to l_q^{\psi(j)}) \leqslant c_2 (2^j)^{ml + Nl(1/q - 1/p)} \alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)),$$

c_2 independent of n and j. Choose $n = [\psi(j)/2]$. Then $(2^j) \sim n^{1/N(l+1)}$ and

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \geqslant c_3 n^{-(l/(l+1))(m/N + 1/q - 1/p)} \alpha_n(l_p^{2n} \to l_q^{2n}), \qquad (2.16)$$

$c_3 > 0$ independent of n. For $p \geqslant q$, $\alpha_n(l_p^{2n} \to l_q^{2n}) = n^{1/q - 1/p}$; cf. [13]. Therefore in this case

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \geqslant c_3 n^{-\gamma}, \qquad \gamma = \frac{l}{l+1} \left(\frac{m}{N} + \frac{1}{l} \left(\frac{1}{p} - \frac{1}{q} \right) \right),$$

which we wanted to show. Consider now $p < q$. The approximation numbers of diagonal maps from l_1 to l_2 are known; cf. [7, 13]. Especially $\alpha_n(l_1^{2n} \to l_2^{2n}) = 1/2^{1/2}$. By duality also $\alpha_n(l_2^{2n} \to l_\infty^{2n}) = 1/2^{1/2}$. But for $1 \leqslant p < q \leqslant 2$

$$\alpha_n(l_p^{2n} \to l_q^{2n}) \geqslant \alpha_n(l_1^{2n} \to l_2^{2n}) = 1/2^{1/2}$$

and for $2 \leqslant p < q \leqslant \infty$

$$\alpha_n(l_p^{2n} \to l_q^{2n}) \geqslant \alpha_n(l_2^{2n} \to l_\infty^{2n}) = 1/2^{1/2}.$$

Using (2.16) we get in these cases of $p < q$

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \geqslant c_4 n^{-\gamma}, \qquad \gamma = \frac{l}{l+1}\left(\frac{m}{N} + \frac{1}{q} - \frac{1}{p}\right).$$

This ends the proof of Theorem 1. ∎

The essential point in the last part, the lower estimate for the approximation numbers, was that Ω contained asymptotically a maximal number of small disjoint balls. More precisely, let Ω satisfy condition $C_k{}^l$ and denote $\Omega^n = \Omega \cap \{x \in \mathbb{R}^N : 2^n < \| x \|_\infty < 2^{n+1}\}$. Let $f(n)$ denote the maximal number of disjoint open cubes of side length $(2^n)^{-l}$ contained in Ω^n. We call Ω a full domain, if

$$\liminf_{n \to \infty} f(n)(2^n)^{-N(1+l)} > 0.$$

In this case, the same method as in the proof works and we have the asymptotic behavior as stated in (c) of Theorem 1.

Theorem 1 does not give the exact order of decay of the approximation numbers for $1 \leqslant p < 2 < q \leqslant \infty$, the same values of p and q for which the approximation numbers of imbeddings over bounded domains seem to be unknown too.

CONJECTURE. If $\Omega \subset \mathbb{R}^N$ is a quasibounded domain fulfilling condition $C_k{}^l$ and $m > N(1/p - 1/q)$ as well as $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, is the order of decay given by

$$\alpha_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-(l/(l+1))(m/N+1/q-1/2)}, \qquad \frac{1}{p} + \frac{1}{q} \leqslant 1,$$

$$\lesssim n^{-(l/(l+1))(m/N+1/2-1/p)}, \qquad \frac{1}{p} + \frac{1}{q} \geqslant 1,$$

with $\lesssim$ being replaced by $\sim$ for full domains?

If this could be shown for bounded domains (i.e., $l = \infty$, $l/(l+1)$ replaced by 1), it would also hold for $C_k{}^l$-quasibounded domains. The same proof as in Theorem 1 applies, with a slightly changed function $\phi(n)$ in part (1).

The Kolmogorov and Gelfand numbers of a continuous linear operator T are generally smaller than the approximation numbers. One has

THEOREM 2. *Let $\Omega \subset \mathbb{R}^N$ be an open set fulfilling condition $C_k{}^l$ with $l \in \mathbb{R}^+$, $k \in \mathbb{N}$ and $1 \leqslant k \leqslant N$. Let $1 \leqslant p, q \leqslant \infty$ with $p > k$. Then*

(a) *For $p \geqslant q$ and $ml > N(1/q - 1/p)$,*

$$\left.\begin{array}{l}\beta_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \\ \gamma_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega))\end{array}\right\} \lesssim n^{-(l/(l+1))(m/N+1/l(1/p-1/q))}.$$

For full domains $\lesssim$ may be replaced by $\sim$.

(b) *For $p \leqslant q$ and $m > N(1/p - 1/q)$,*

$$\left.\begin{array}{l}\beta_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \\ \gamma_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega))\end{array}\right\} \lesssim n^{-(l/(l+1))(m/N+(1/q-1/p))}.$$

For full domains and

$$1 \leqslant p \leqslant q \leqslant 2 \quad \textit{the order is exact for } \beta_n\,,$$

$$2 \leqslant p \leqslant q \leqslant \infty \quad \textit{the order is exact for } \gamma_n\,.$$

The proof is similar to the one of Theorem 1 and is therefore omitted.

CONJECTURE. For quasibounded domains fulfilling condition $C_k{}^l$, does one have

$$1 \leqslant p \leqslant q \leqslant 2: \qquad \gamma_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-(l/(l+1))(m/N)};$$

$$2 \leqslant p \leqslant q \leqslant \infty: \qquad \beta_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-(l/(l+1))(m/N)},$$

$$1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty: \beta_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-(l/(l+1))(m/N+\frac{1}{2}-1/p)},$$

$$1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty: \gamma_n(\mathring{W}_p{}^m(\Omega) \to L_q(\Omega)) \lesssim n^{-(l\ (l+1))(m/N+1/q-\frac{1}{2})}\ ?$$

These orders would be optimal for a bounded domain (with $l/(l+1)$ being replaced by 1), as some easy lower estimates for $\beta_n(l_p^{2n} \to l_q^{2n})$ and $\gamma_n(l_p^{2n} \to l_q^{2n})$ resulting from known values for these numbers in sequence spaces show: cf. for these [7, 13]. Again a proof of the conjecture in the case of a bounded domain would be sufficient to get the result for $C_k{}^l$-quasibounded domains. A proof similar to that for Theorem 1 would apply, with a modified function $\phi(n)$ in part 1 (a different "cut-off point" in the application of the Poincare-type inequality is necessary). For full domains, the result would be optimal.

3. APPLICATION

Let Ω be a quasibounded domain in $\mathbb{R}^N$ fulfilling condition $C_1{}^l$ for some $l > 0$. Let

$$A(x, D) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, D^\alpha$$

be a formally selfadjoint, uniformly strongly elliptic (cf. [3]) differential operator of order $2m$ with real coefficients $a_\alpha \in C^\infty(\Omega)$ that are uniformly continuous for $|\alpha| = 2m$ and measurable and uniformly bounded for $|\alpha| \leqslant 2m$. Define

$$T: \mathscr{D}(T) \subset L_2(\Omega) \to L_2(\Omega)$$

by

$$\mathscr{D}(T) := \mathring{W}_2{}^m(\Omega) \cap \{f \in L_2(\Omega): A(x, D) f \in L_2(\Omega)\}$$

and

$$Tf := A(x, D) f, \qquad f \in \mathscr{D}(T)$$

(Dirichlet boundary conditions).

THEOREM 3. *T is a closed linear operator the spectrum of which consists of denuperably many eigenvalues. If they are ordered by increasing absolute value and counted according to their multiplicity, denoted by $\lambda_n(T)$, we have*

$$|\lambda_n(T)| \gtrsim n^{(l/(l+1))(2m/N)}.$$

Moreover, if Ω is a full domain fulfilling $C_1{}^l$,

$$|\lambda_n(T)| \sim n^{(l/(l+1))(2m/N)}.$$

Proof. The first statement was proved by Clark [5]. Using standard techniques of the theory of elliptic operators (cf. [3]) it was shown in [10] that there is $\lambda \in \mathbb{R} \sim \{0\}$ and an injective, positive, compact, selfadjoint operator C: $L_2(\Omega) \to L_2(\Omega)$, which can be considered also as continuous linear map $\bar{C}$: $L_2(\Omega) \to \mathring{W}_2{}^m(\Omega)$ such that the following relation between the eigenvalues of T and C holds,

$$|\lambda_n(C)| = |(\lambda_n(T) + \lambda)|^{-1/2}, \qquad n \in \mathbb{N}. \tag{3.1}$$

Further, $C_0{}^\infty(\Omega) \subset \mathscr{D}(T) \subset \operatorname{Im} C$. Since C is a selfadjoint operator in $L_2(\Omega)$, the approximation numbers of C are just given by

$$\alpha_n(C) = |\lambda_n(C)|;$$

cf. Pietsch [11]. But $C = I \circ \bar{C}$, where I: $\mathring{W}_2{}^m(\Omega) \to L_2(\Omega))$ is the Sobolev imbedding. Hence by Theorem 1, $p = q = 2$,

$$\begin{aligned} |\lambda_n(T) + \lambda|^{-1/2} = |\lambda_n(C)| &= \alpha_n(C) \\ &\leqslant \|\bar{C}\| \, \alpha_n(I) \\ &\lesssim n^{-(l/(l+1))(m/N)}. \end{aligned}$$

Therefore $|\lambda_n(T)| \gtrsim n^{(l/(l+1))(2m/N)}$. It is an easy consequence of (4.3) and (4.11) of [10] that $\bar{C}: L_2(\Omega) \to \operatorname{Im} \bar{C} \subset \mathring{W}_2^m(\Omega)$ is an isomorphism onto $\operatorname{Im} \bar{C}$. By the properties of the approximation numbers,

$$\begin{aligned} \alpha_n(\mathrm{Id}: C_0^\infty(\Omega) \to L_2(\Omega)) &\leqslant \alpha_n(\mathrm{Id}: \operatorname{Im} \bar{C} \to L_2(\Omega)) \\ &\leqslant \| \bar{C}^{-1}: \operatorname{Im} \bar{C} \to L_2(\Omega) \| \, \alpha_n(C: L_2(\Omega) \to L_2(\Omega)) \\ &= \| \bar{C}^{-1} \| \, | \lambda_n(C) | = \| \bar{C}^{-1} \| \, | \lambda_n(T) + \lambda |^{-1/2} \end{aligned} \tag{3.2}$$

where $C_0^\infty(\Omega)$ and $\operatorname{Im} \bar{C}$ are equipped with the $\mathring{W}_2^m$-norm. If Ω is a full domain fulfilling C_1^l,

$$\alpha_n(\mathrm{Id}: C_0^\infty(\Omega) \to L_2(\Omega)) \gtrsim n^{-(l/(l+1))(m/N)}.$$

This follows exactly as in part (2) of the proof of Theorem 1. Hence by (3.2)

$$| \lambda_n(T) | \lesssim n^{(l/(l+1))(2m/N)}. \quad \blacksquare$$

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